Topological applications of Wadge theory II

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Reasonably closed Wadge classes

Given $i \in 2$, set:

$$Q_i = \{x \in 2^\omega : x(n) = i \text{ for all but finitely many } n \in \omega\}$$

Notice that every element of $2^\omega \setminus (Q_0 \cup Q_1)$ is obtained by alternating finite blocks of zeros and finite blocks of ones.

Define the function $\phi : 2^\omega \setminus (Q_0 \cup Q_1) \longrightarrow 2^\omega$ by setting

$$\phi(x)(n) = \begin{cases} 0 & \text{if the } n^{\text{th}} \text{ block of zeros of } x \text{ has even length} \\ 1 & \text{otherwise} \end{cases}$$

where we start counting with the 0^{th} block of zeros. It is easy to check that ϕ is continuous.

Definition (Steel, 1980)

Let $\mathbf{\Gamma}$ be a Wadge class in 2^ω. We will say that $\mathbf{\Gamma}$ is *reasonably closed* if $\phi^{-1}[A] \cup Q_0 \in \mathbf{\Gamma}$ for every $A \in \mathbf{\Gamma}$.

Why would anybody need that?

Lemma (Harrington)

Let $\mathbf{\Gamma} = [B]$ *be a reasonably closed Wadge class in* 2^ω*. If* $A \leq B$ *then this is witnessed by an injective function.*

The above lemma will be useful to us because every injective continuous function $f : 2^\omega \longrightarrow 2^\omega$ is an embedding.

Proof.

Let $A^* = \phi^{-1}[A] \cup Q_0$. Since $\mathbf{\Gamma}$ is reasonably closed, we can fix $\sigma : 2^{<\omega} \longrightarrow 2^{<\omega}$ such that $f_\sigma : 2^\omega \longrightarrow 2^\omega$ witnesses $A^* \leq B$. We will construct $\tau : 2^{<\omega} \longrightarrow 2^{<\omega}$ such that $f_\tau : 2^\omega \longrightarrow 2^\omega$ witnesses $A \leq A^*$ and $f_\sigma \circ f_\tau$ is injective.

Make sure that

1. $\tau(s)$ always ends with a 1
2. There are exactly $|s|$ blocks of zeros in $\tau(s)$
3. $s(n)$ is the parity of the n^{th} block of zeros in $\tau(s)$

Begin by setting $\tau(\varnothing) = \langle 1 \rangle$.
Given $s \in 2^{<\omega}$, notice that $\tau(s)^\frown\vec{0} \in A^*$ and $\tau(s)^\frown\vec{1} \notin A^*$.
Since f_σ witnesses that $A^* \leq B$, we must have $f_\sigma(\tau(s)^\frown\vec{0}) \in B$ and $f_\sigma(\tau(s)^\frown\vec{1}) \notin B$. Therefore, we can find $k \in \omega$ such that

$$\sigma(\tau(s)^\frown 0^k) \neq \sigma(\tau(s)^\frown 1^k)$$

Now simply pick $\tau(s^\frown i) \supseteq \tau(s)^\frown i^k$ for $i = 0, 1$ satisfying conditions (1), (2) and (3).
To check that f_τ has the desired properties, observe that

- $\mathrm{ran}(f_\tau) \subseteq 2^\omega \setminus (Q_0 \cup Q_1)$ (By conditions 1 and 2)
- $\phi(f_\tau(x)) = x$ for every $x \in 2^\omega$ (By conditions 1 and 3)

□

Our main tool: Steel's theorem

Given a Wadge class $\boldsymbol{\Gamma}$ in 2^ω and $X \subseteq 2^\omega$, we will say that X is *everywhere properly* $\boldsymbol{\Gamma}$ if $X \cap [s] \in \boldsymbol{\Gamma} \setminus \check{\boldsymbol{\Gamma}}$ for every $s \in 2^{<\omega}$.

Theorem (Steel, 1980)

Let $\boldsymbol{\Gamma}$ be a reasonably closed Wadge class in 2^ω. Assume that X and Y are subsets of 2^ω that satisfy the following:

- *X and Y are everywhere properly $\boldsymbol{\Gamma}$*
- *X and Y are either both meager or both comeager*

Then there exists a homeomorphism $h : 2^\omega \longrightarrow 2^\omega$ such that $h[X] = Y$.

Proof.

Without loss of generality, fix closed nowhere dense subsets X_n and Y_n of 2^ω for $n \in \omega$ such that $X \subset \bigcup_{n\in\omega} X_n$ and $Y \subset \bigcup_{n\in\omega} Y_n$. We will combine Harrington's Lemma with Knaster-Reichbach systems. (To be continued...)

Knaster-Reichbach covers

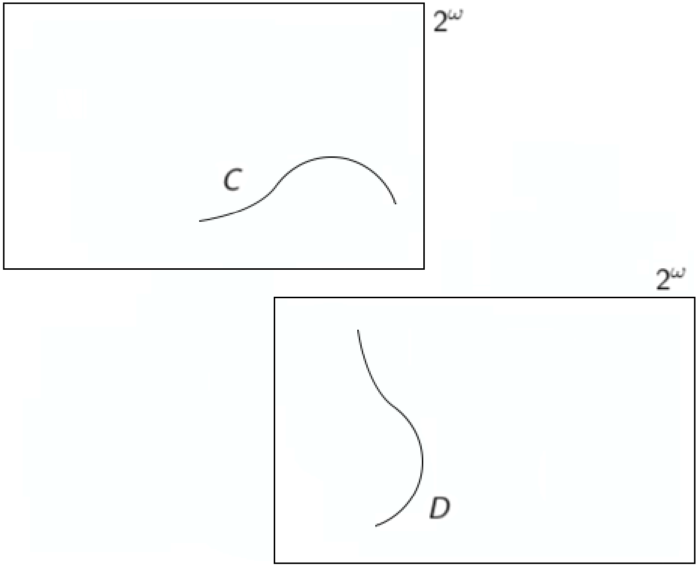
Fix a homeomorphism $h : C \longrightarrow D$ between closed nowhere dense subsets of 2^ω. We will say that $\langle \mathcal{U}, \mathcal{V}, \psi \rangle$ is a *Knaster-Reichbach cover* (briefly, a KR-cover) for $\langle 2^\omega \setminus C, 2^\omega \setminus D, h \rangle$ if the following conditions hold:

- $\mathcal{U}$ is a cover of $2^\omega \setminus C$ consisting of pairwise disjoint non-empty clopen subsets of 2^ω
- $\mathcal{V}$ is a cover of $2^\omega \setminus D$ consisting of pairwise disjoint non-empty clopen subsets of 2^ω
- $\psi : \mathcal{U} \longrightarrow \mathcal{V}$ is a bijection
- If $f : 2^\omega \longrightarrow 2^\omega$ is a bijection such that $h \subseteq f$ and $f[U] = \psi(U)$ for every $U \in \mathcal{U}$ (we say that f *respects* ψ), then f is continuous on C and f^{-1} is continuous on D

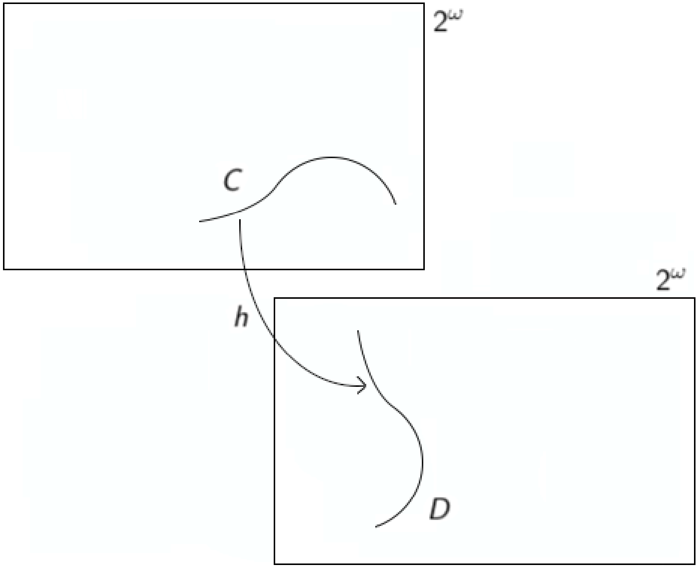
Lemma (see Medini, 2015)

Let $h : C \longrightarrow D$ be a homeomorphism between closed nowhere dense subsets of 2^ω. Then there exists a KR*-cover for $\langle 2^\omega \setminus C, 2^\omega \setminus D, h \rangle$.*

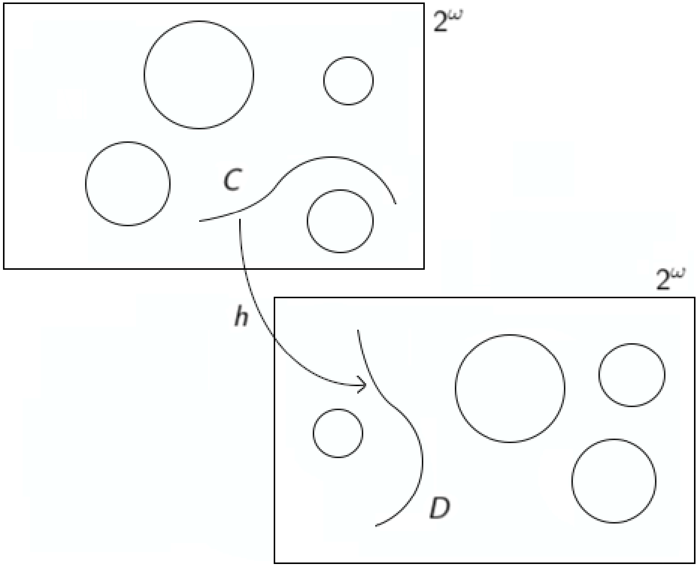
Knaster-Reichbach covers



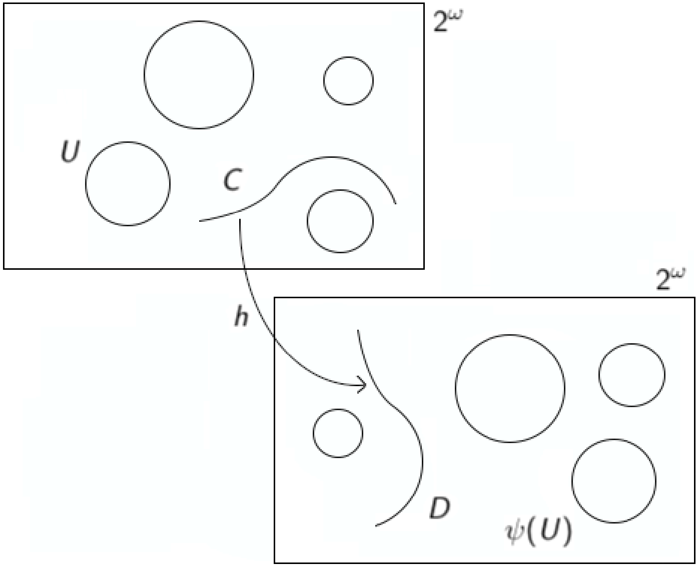
Knaster-Reichbach covers



Knaster-Reichbach covers



Knaster-Reichbach covers

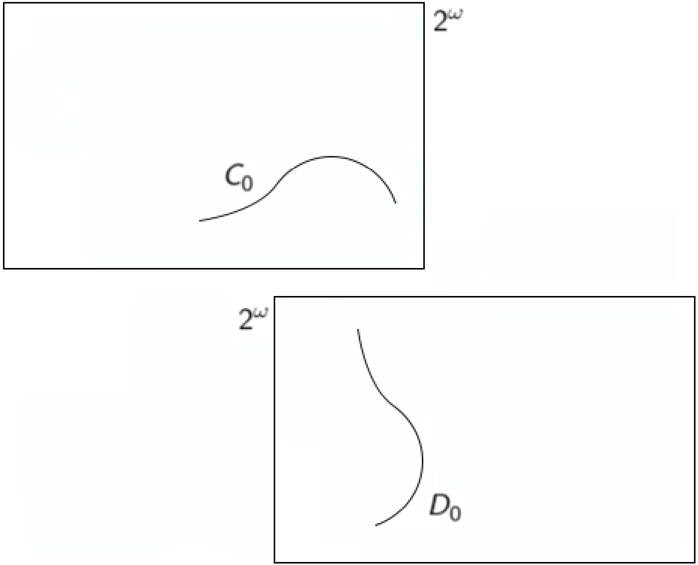


Knaster-Reichbach systems

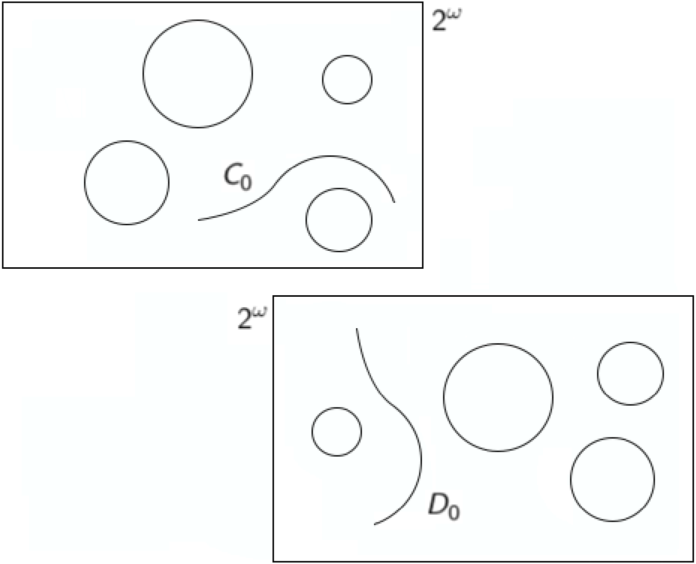
Fix an admissible metric on 2^ω. We will say that a sequence $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$ is a *Knaster-Reichbach system* (briefly, a KR-system) if the following conditions are satisfied:

- Each $h_n : C_n \longrightarrow D_n$ is a homeomorphism between closed nowhere dense subsets of 2^ω
- $h_m \subseteq h_n$ whenever $m \leq n$
- Each $\mathcal{K}_n = \langle \mathcal{U}_n, \mathcal{V}_n, \psi_n\rangle$ is a KR-cover for $\langle 2^\omega \setminus C_n, 2^\omega \setminus D_n, h_n\rangle$
- $\mathrm{mesh}(\mathcal{U}_n) \leq 2^{-n}$ and $\mathrm{mesh}(\mathcal{V}_n) \leq 2^{-n}$ for each n
- $\mathcal{U}_m$ refines $\mathcal{U}_n$ and $\mathcal{V}_m$ refines $\mathcal{V}_n$ whenever $m \geq n$
- Given $U \in \mathcal{U}_m$ and $V \in \mathcal{U}_n$ with $m \geq n$, then $U \subseteq V$ if and only if $\psi_m(U) \subseteq \psi_n(V)$

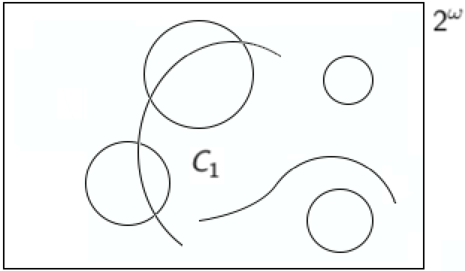
Knaster-Reichbach systems

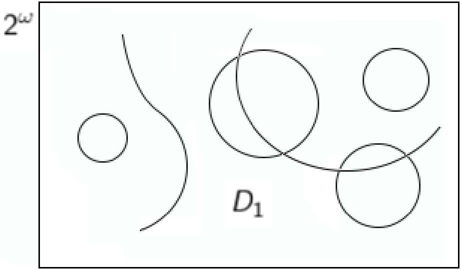


Knaster-Reichbach systems

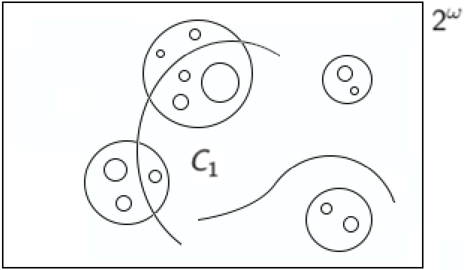


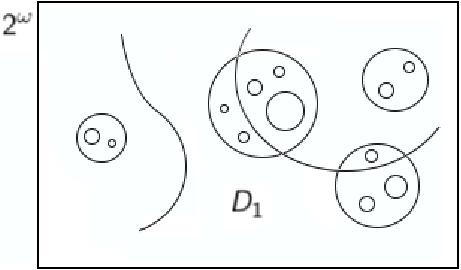
Knaster-Reichbach systems





Knaster-Reichbach systems





Why do we care about Knaster-Reichbach systems?

Because they give us homeomorphisms!

Theorem (see Medini, 2015)

Assume that $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$ *is a* KR*-system. Then there exists a homeomorphism* $h : 2^\omega \longrightarrow 2^\omega$ *such that* $h \supseteq \bigcup_{n\in\omega} h_n$*.*

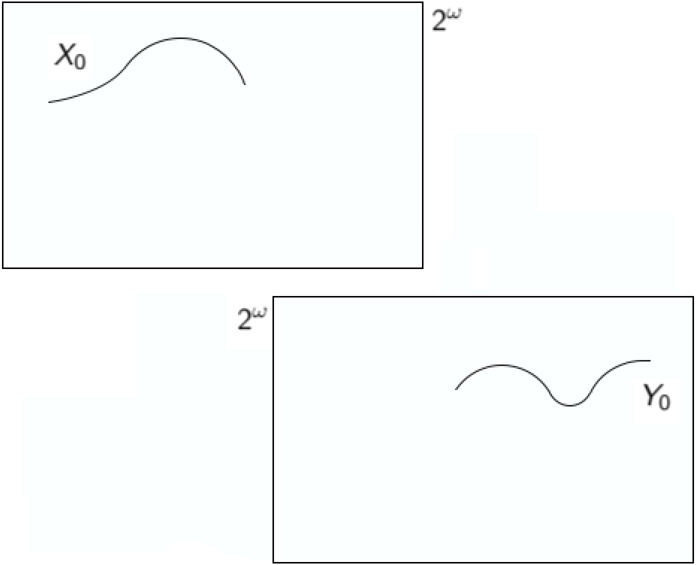
Corollary

Let X *and* Y *be subspaces of* 2^ω*. Assume that* $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$ *is a* KR*-system satisfying the following additional conditions:*

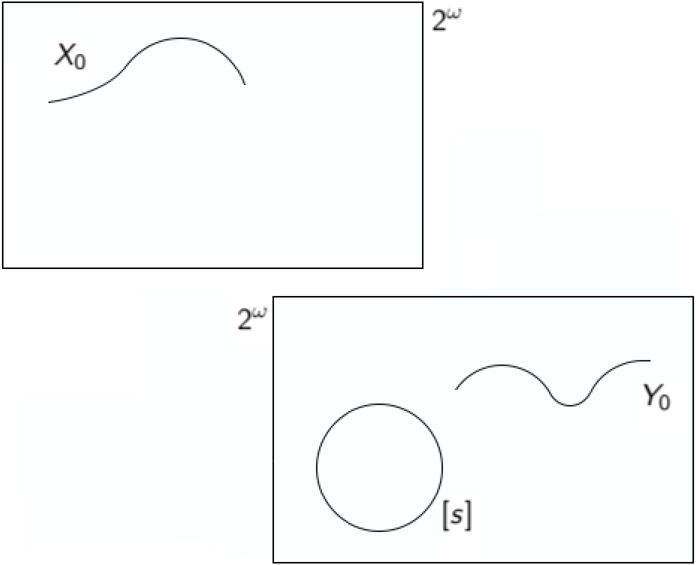
- $X \subseteq \bigcup_{n\in\omega} C_n$
- $Y \subseteq \bigcup_{n\in\omega} D_n$
- $h_n[X \cap C_n] = Y \cap D_n$ *for each* n

Then there exists a homeomorphism $h : 2^\omega \longrightarrow 2^\omega$ *such that* $h \supseteq \bigcup_{n\in\omega} h_n$ *and* $h[X] = Y$*.*

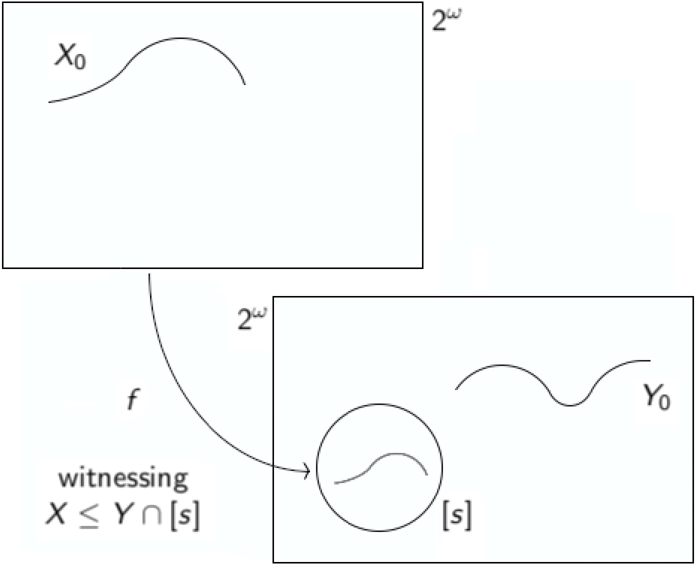
Proof of Steel's theorem



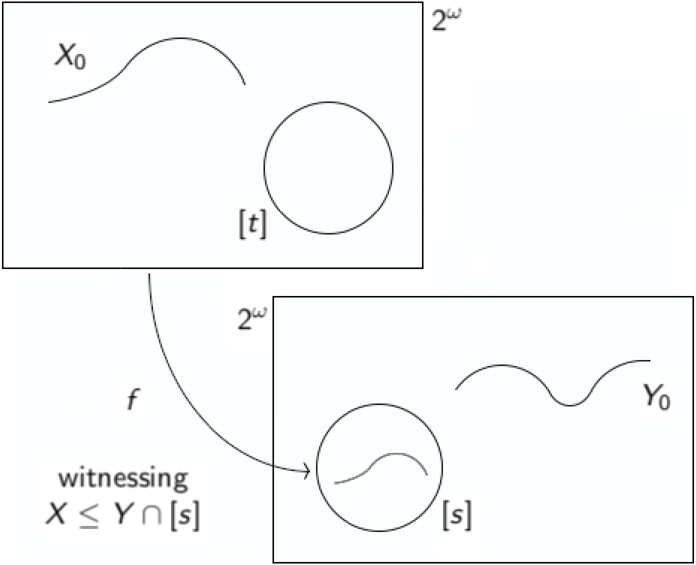
Proof of Steel's theorem



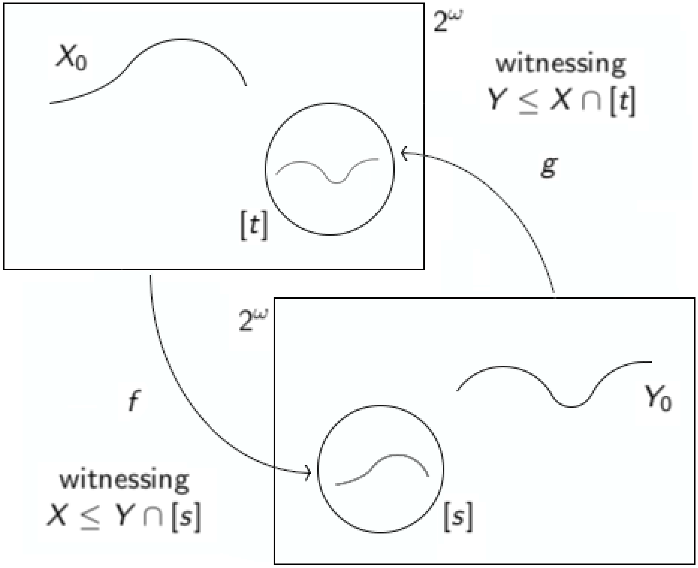
Proof of Steel's theorem



Proof of Steel's theorem



Proof of Steel's theorem



Proof of Steel's theorem

Remember that our strategy is to construct a KR-system $\langle\langle h_n, \mathcal{K}_n\rangle : n \in \omega\rangle$. We have seen how to begin:

- $C_0 = X_0 \cup g[Y_0]$
- $D_0 = Y_0 \cup f[X_0]$
- $h_0 = (f \restriction X_0) \cup (g^{-1} \restriction g[Y_0])$

Then obtain a KR-cover $\langle \mathcal{U}_0, \mathcal{V}_0, \psi_0\rangle$ for $\langle 2^\omega \setminus C_0, 2^\omega \setminus D_0, h_0\rangle$.

The next step is like the first one, but with the following changes:

- Instead of working between 2^ω and 2^ω, work between U and $\psi_0(U)$, where $U \in \mathcal{U}_0$
- Instead of looking at X_0 and Y_0, look at $X_1 \cap U$ and $Y_1 \cap \psi_0(U)$
- Repeat for every $U \in \mathcal{U}_0$, then union up the partial homeomorphisms to get h_1

Keep going like this for ω more steps...

□

Thank you for your attention

and have a good evening!